ELEN E3401: Electromagnetics

Spring 2025

Prof. Keren Bergman

Lecture #12

COLUMBIA ENGINEERING

The Fu Foundation School of Engineering and Applied Science

Maxwell's Equations: statics

<u>Electrostatics</u>:

$$\vec{\nabla} \cdot \vec{D} = \rho_V$$

$$\vec{\nabla} \times \vec{E} = 0$$

<u>Magnetostatics</u>:

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{H} = \vec{J}$$

Conductors

Free electrons loosely attached to atoms.
With applied $\vec{E}$ external field, electrons migrate in opposite to field direction

Conduction current: $\vec{J} = \sigma\vec{E}$ (A/m^2) Ohm's Law

Silver:	6.2×10^7	[S/m]	[σ increases with decreasing temperature]
Gold:	4.1×10^7		
Germanium:	2.2	}	Semiconductors → need to dope
Silicon:	4.4×10^{-4}	}	
Glass:	10^{-12}	→	dielectric

Conductors

Perfect dielectric $\sigma = 0$ Perfect conductor $\sigma = \infty$

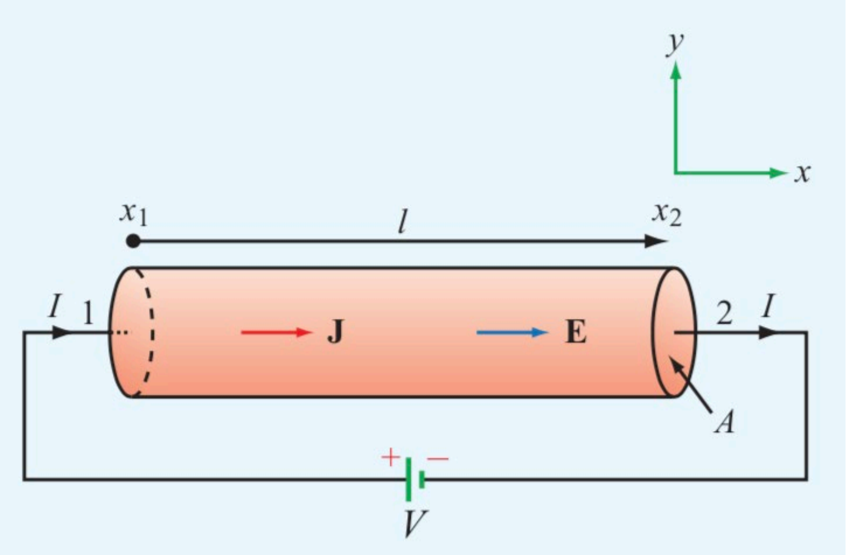
$\vec{J} = \sigma\vec{E}$ (A/m^2) Ohm's Law

$$\begin{cases} \text{With } \sigma = 0 \text{ (dielectric), } \vec{J} = 0 \text{ (independent of } \vec{E}\text{)} \\ \text{With } \sigma = \infty,\ \vec{E} = \dfrac{\vec{J}}{\sigma} = 0 \end{cases}$$

Perfect dielectric $\longrightarrow$ $\vec{J} = 0$ Perfect conductor (Equipotential) $\longrightarrow$ $\vec{E} = 0$

$$V_{21} = -\int_{P_1}^{P_2} \vec{E} \cdot d\vec{l} = V_2 - V_1 = 0$$

Resistance



Conductor length: $l = x_2 - x_1$
Cross section area: A

$$\vec{E} = \hat{x}E_x$$

Higher potential point 1
Lower potential point 2

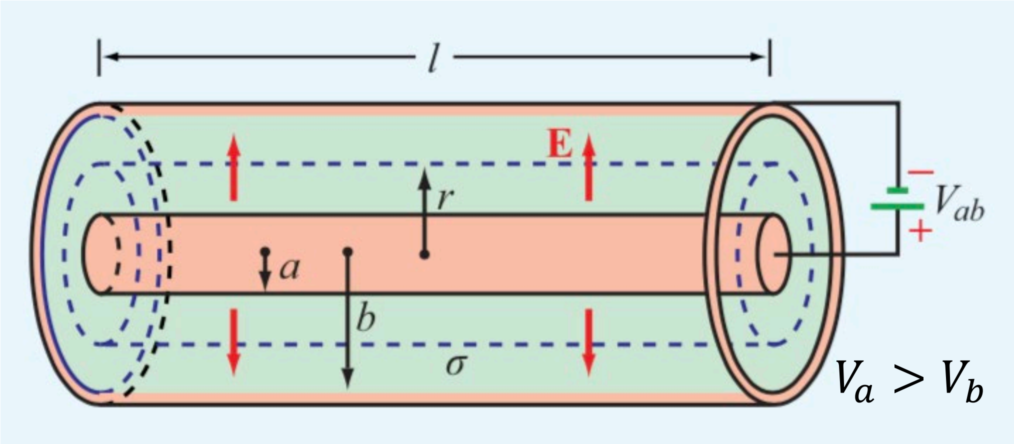
$$V = V_1 - V_2 = -\int_{x_2}^{x_1} \vec{E} \cdot d\vec{l} = -\int_{x_2}^{x_1} \hat{x}E_x \cdot \hat{x}dl = E_x l$$

$$I = \int_A \vec{J} \cdot d\vec{s} = \int_A \sigma\vec{E} \cdot d\vec{s} = \sigma E_x A \qquad R = \frac{V}{I} = \frac{E_x l}{\sigma E_x A} = \frac{l}{\sigma A}\ [\Omega]$$

Can generalize result for resistance

$$R = \frac{V}{I} = \frac{-\int_l \vec{E} \cdot d\vec{l}}{\int_S \vec{J} \cdot d\vec{s}} = \frac{-\int_l \vec{E} \cdot d\vec{l}}{\int_S \sigma\vec{E} \cdot d\vec{s}} \qquad R = \frac{l}{\sigma A}\ [\Omega] \qquad G = \frac{1}{R} = \frac{\sigma A}{l}\ [S]$$

Example - coax cable



Conductance of coax

Voltage inner conductor > outer

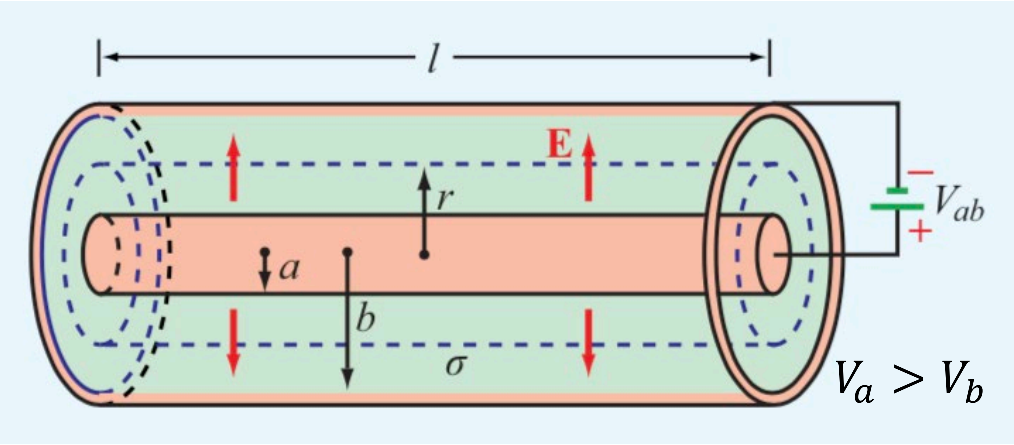
Coax cable length, l

Inner radius = a
Outer radius = b

Insulation material has conductivity, σ

Obtain G'/ unit length of insulation layer

Example - coax cable



<u>Conductance of coax</u>

Voltage inner conductor > outer

Coax cable length, l

Inner radius = a

Outer radius = b

Insulation conductivity, σ

<u>Obtain</u> G'/ unit length of <u>insulation layer</u>

I = current flowing from inner conductor to outer conductor through insulation → $\hat{r}$

Area through which current flows: $\underbrace{\mathrm{A} = 2\pi r l}_{\text{cylindrical}}$

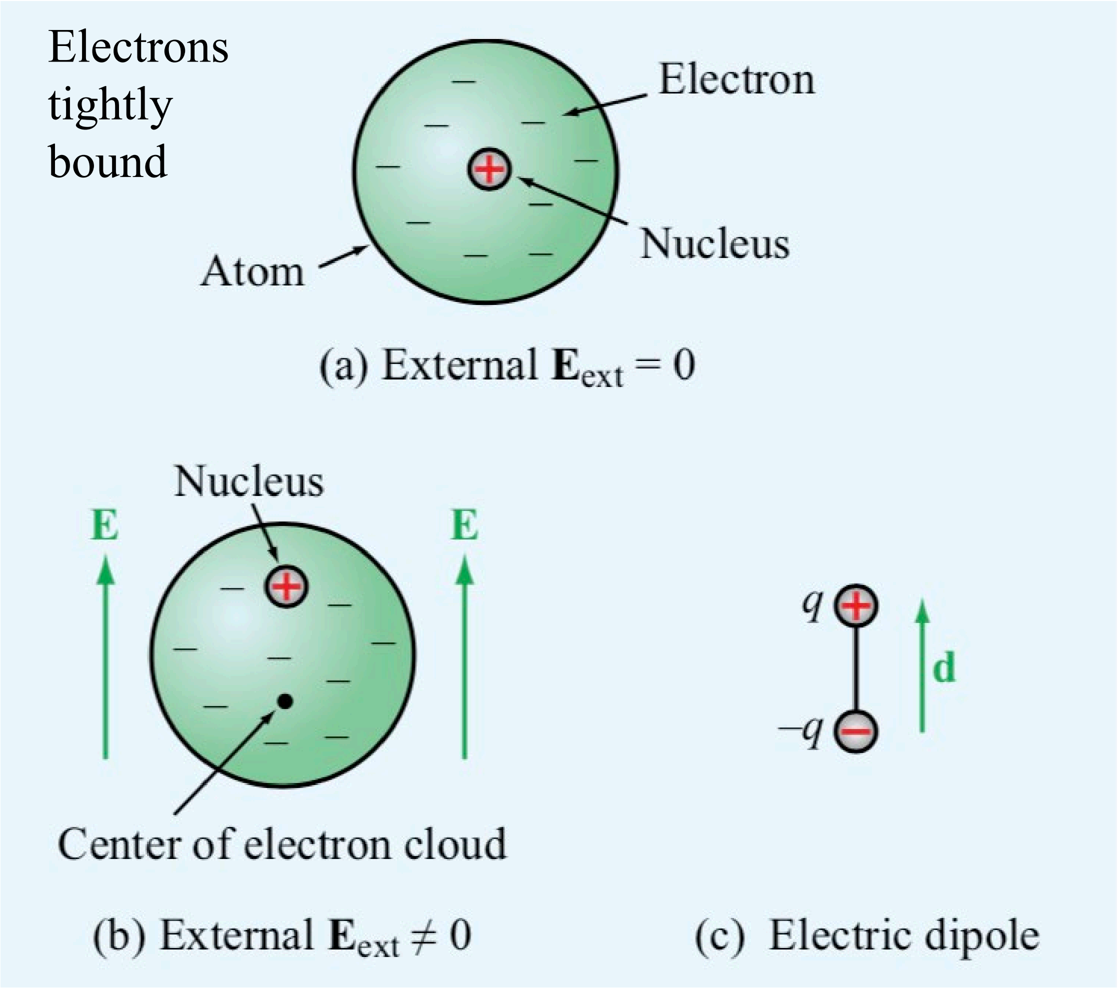
$$\vec{J} = \hat{r}\frac{I}{A} = \hat{r}\frac{I}{2\pi r l} \qquad \vec{J} = \sigma\vec{E} \qquad \vec{E} = \hat{r}\frac{I}{2\pi\sigma r l} \text{ from inner to outer}$$

$$V_{ab} = -\int_b^a \vec{E}\cdot d\vec{l} = -\int_b^a \hat{r}\frac{I}{2\pi\sigma r l}\cdot\hat{r}dr = \frac{I}{2\pi\sigma l}\ln\left(\frac{b}{a}\right)$$

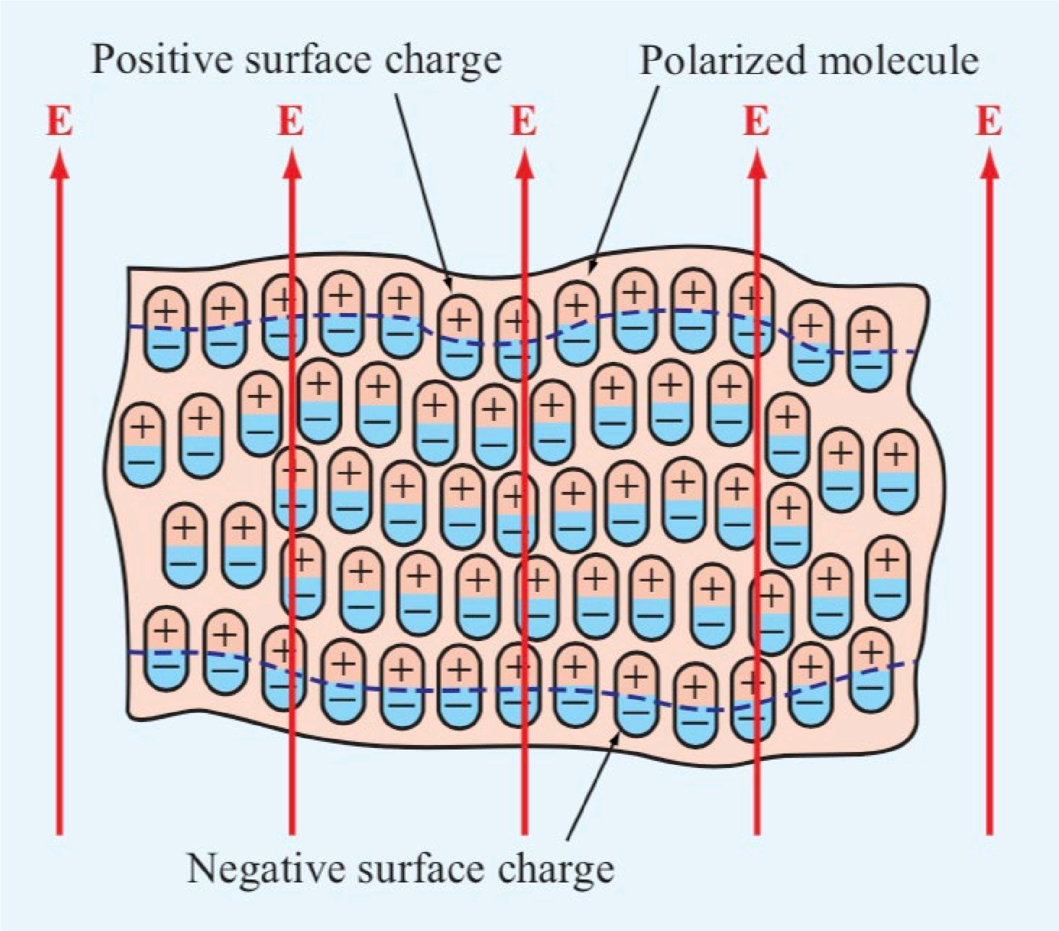
$$G' = \frac{G}{l} = \frac{1}{Rl} = \frac{I}{V_{ab}l} = \frac{2\pi\sigma}{\ln(b/a)} \ [S/m]$$

Same as table 2.1

Dielectrics – dipole model



Dielectrics



→ Can have polar materials with <u>permanent dipole moments</u> that are either random or aligned with $\vec{E}$ (liquid crystals)

Electric Breakdown

The dielectric strength E_{ds} is the largest magnitude of **E** *that the material can sustain without breakdown.*

Table 4-2: Relative permittivity (dielectric constant) and dielectric strength of common materials.

Material	Relative Permittivity, ε_r	Dielectric Strength, E_{ds} (MV/m)
Air (at sea level)	1.0006	3
Petroleum oil	2.1	12
Polystyrene	2.6	20
Glass	4.5–10	25–40
Quartz	3.8–5	30
Bakelite	5	20
Mica	5.4–6	200

$\varepsilon = \varepsilon_r \varepsilon_0$ and $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m.

Polarization field

Free space $\vec{D} = \epsilon_0 \vec{E}$

In a dielectric we define: $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ ← Electric polarization field

Linear – $\vec{P}$ is linear with $\vec{E}$

Isotropic – $\vec{P}$ and $\vec{E}$ in same direction

Anisotropic – $\vec{P}$ and $\vec{E}$ may have different directions

Homogeneous - ϵ, μ, σ are uniform constant

Electric susceptibility

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

$$\vec{D} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E} = \epsilon_0 (1 + \chi_e) \vec{E} = \epsilon \vec{E}$$

$$\epsilon = \epsilon_0 (1 + \chi_e) \qquad \epsilon_r = \frac{\epsilon}{\epsilon_0}$$

Electric boundary conditions

Gauss $\vec{\nabla} \cdot \vec{D} = \rho_V$ → $\oint_S \vec{D} \cdot d\vec{s} = Q$

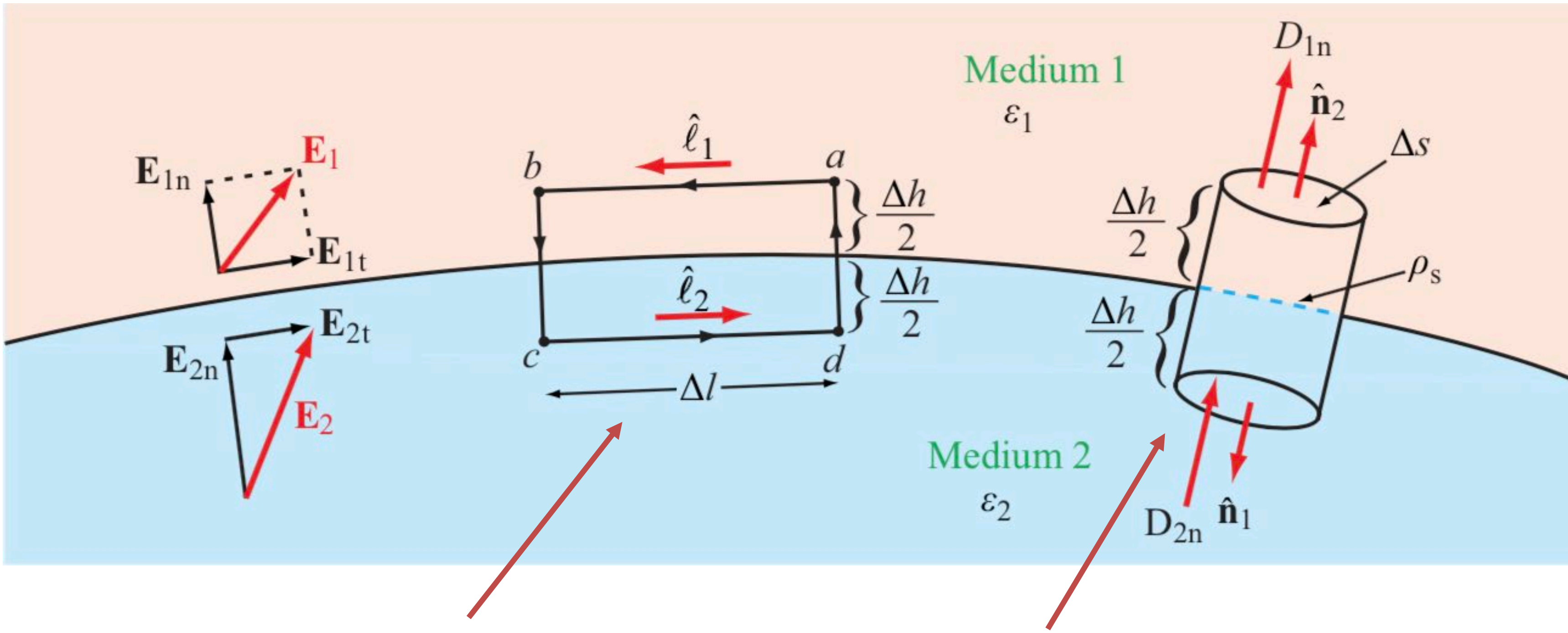
Kirchoff (Faraday): $\vec{\nabla} \times \vec{E} = 0$ → $\oint_C \vec{E} \cdot d\vec{l} = 0$

From Maxwell's equations – obtains set of boundary conditions on:
$\vec{E}$, $\vec{D}$, and $\vec{J}$ at interface of any 2 media

(later will do $\vec{H}$, $\vec{B}$)

Derived from electrostatics → apply when $\frac{\partial}{\partial t} \neq 0$

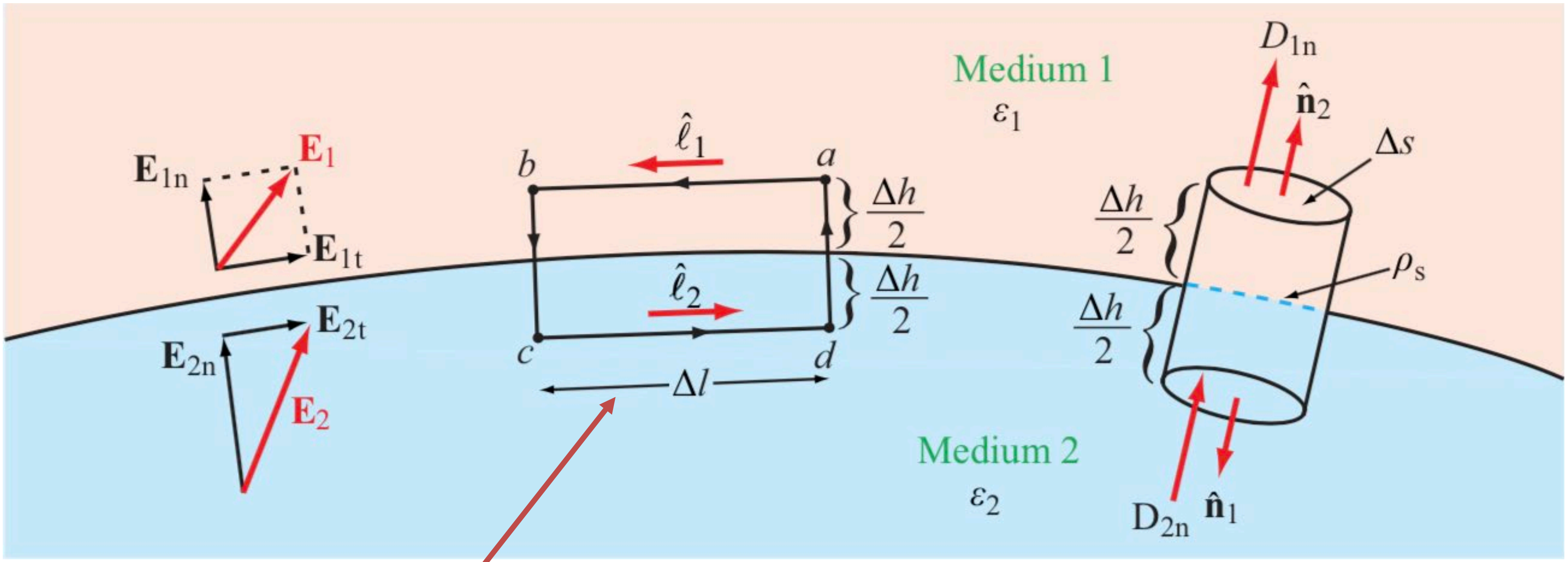
Tangential/normal components of $\vec{E}$ and $\overrightarrow{D}$



Faraday: $\vec{\nabla} \times \vec{E} = 0 \rightarrow \oint_C \vec{E} \cdot d\vec{l} = 0$

Gauss: $\vec{\nabla} \cdot \overrightarrow{D} = \rho_V \rightarrow \oint_S \overrightarrow{D} \cdot d\vec{s} = Q$

Tangential components of $\vec{E}$ and $\vec{D}$



Consider closed rectangular loop **a-b-c-d-a**

Conservative field: $\oint_C \vec{E} \cdot d\vec{l} = 0$ Line integral around closed path is zero

Let $\Delta h \to 0$ then $\overline{bc}$ and $\overline{da} \to 0$

$$\oint_C \vec{E} \cdot d\vec{l} = \int_a^b \vec{E}_1 \cdot \hat{l}_1 dl + \int_c^d \vec{E}_2 \cdot \hat{l}_2 dl = 0$$

Medium 1 Medium 2

Tangential components of $\vec{E}$ and $\vec{D}$

$$\left.\begin{aligned}\vec{E}_1 &= \vec{E}_{1t} + \vec{E}_{1n} \\ \vec{E}_2 &= \vec{E}_{2t} + \vec{E}_{2n}\end{aligned}\right\}$$ Tangential and normal components to the boundary

$\hat{l}_1 = -\hat{l}_2 \qquad (\vec{E}_1 - \vec{E}_2) \cdot \hat{l}_1 = 0$

To satisfy Faraday's law (closed contour, conservative field):
Component of $\vec{E}_1$ along $\hat{l}_1$must equal component of $\vec{E}_2$ along $\hat{l}_1$ for all $\hat{l}_1$ tangential to the boundary

$$\boxed{\vec{E}_{1t} = \vec{E}_{2t}} \qquad \text{[V/m]}$$

Tangential components of $\vec{E}$ and $\vec{D}$

Tangential component of the $\vec{E}$ field is <u>continuous</u> across boundary for <u>any</u> 2 media

$$\vec{D}_1 = \epsilon_1 \vec{E}_{1t} + \epsilon_1 \vec{E}_{1n}$$

$$\vec{D}_2 = \epsilon_2 \vec{E}_{2t} + \epsilon_2 \vec{E}_{2n}$$

$\Rightarrow$

$$\vec{D}_{1t} = \epsilon_1 \vec{E}_{1t}$$

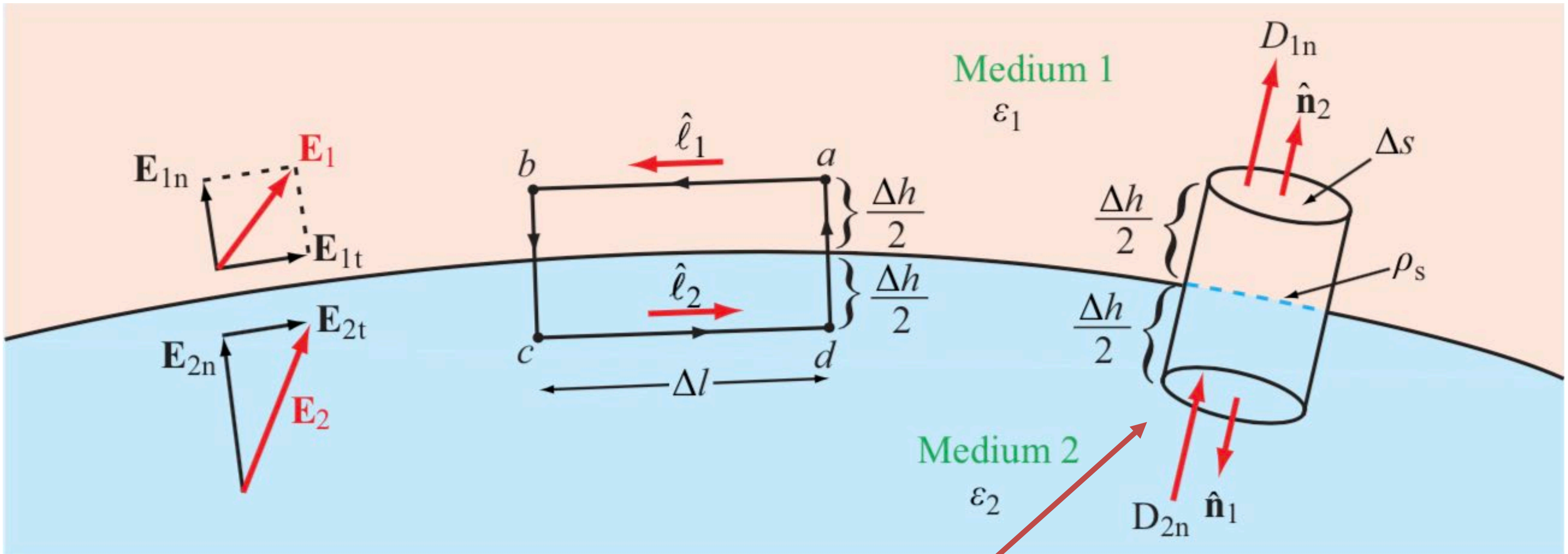
$$\vec{D}_{2t} = \epsilon_2 \vec{E}_{2t}$$

$$\vec{D}_{2t} = \epsilon_2 \vec{E}_{1t}$$

(since $\vec{E}_{1t} = \vec{E}_{2t}$)

$$\frac{\vec{D}_{1t}}{\epsilon_1} = \frac{\vec{D}_{2t}}{\epsilon_2} \qquad \frac{\vec{D}_{1t}}{\vec{D}_{2t}} = \frac{\epsilon_1}{\epsilon_2}$$

Normal components of $\vec{E}$ and $\vec{D}$



Apply Gauss's Law: total outward flux through cylinder must equal total charge inside.

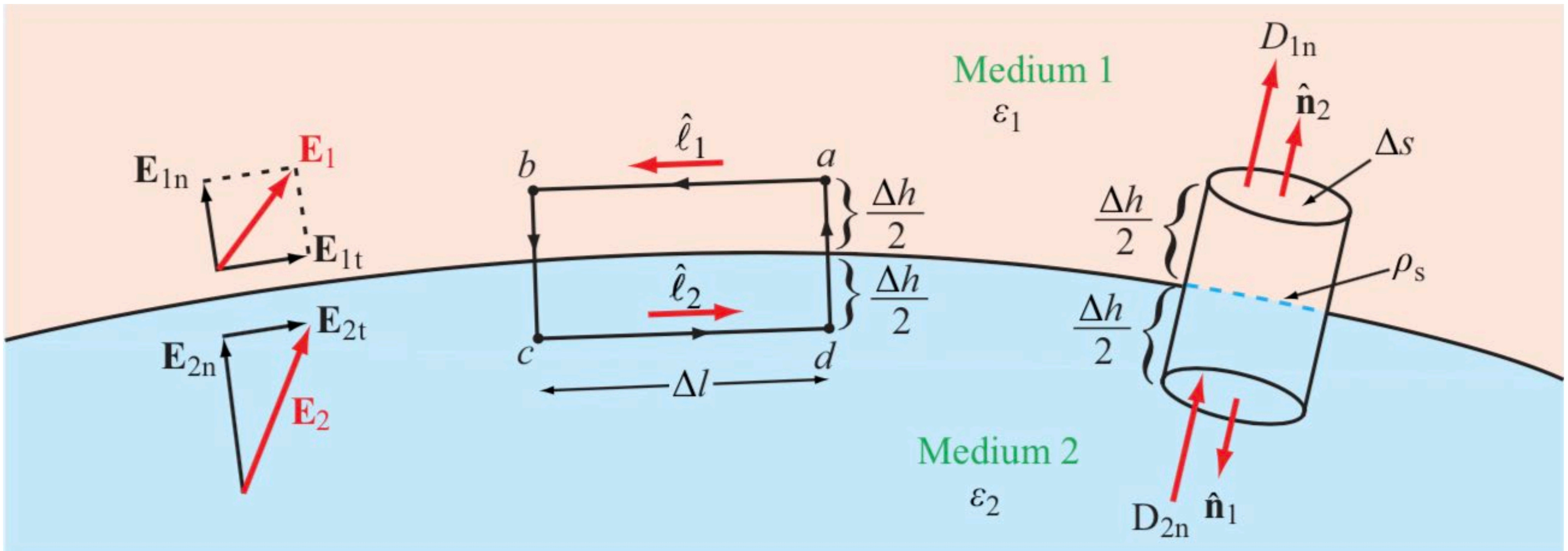
$$\oint_S \vec{D} \cdot d\vec{s} = Q \qquad \vec{\nabla} \cdot \vec{D} = \rho_V$$

Let cylinder height $\Delta h \to 0$ → only flux is from top/bottom surfaces.

$$Q = \rho_s \Delta s \qquad \text{Surface charge density} \times \text{surface differential}$$

$$\oint_S \vec{D} \cdot d\vec{s} = \int_{top} \vec{D}_1 \cdot \hat{n}_2 ds + \int_{bottom} \vec{D}_2 \cdot \hat{n}_1 ds = \rho_s \Delta s$$

Normal components of $\vec{E}$ and $\vec{D}$



$\hat{n}_1$ and $\hat{n}_2$ are outward normal unit vectors from the surface

$$\hat{n}_1 = -\hat{n}_2$$

$$\hat{n}_2 \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s \qquad [\text{C/m}^2]$$

$$\boxed{D_{1n} - D_{2n} = \rho_s} \qquad [\text{C/m}^2]$$

Normal components of $\vec{D}$ change by surface charge density

$$\hat{n}_2 \cdot (\epsilon_1 \vec{E}_1 - \epsilon_2 \vec{E}_2) = \rho_s \qquad \boxed{\epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s}$$

Summary

Conservative property of $\vec{E}$:

$$\vec{\nabla} \times \vec{E} = 0 \longrightarrow \oint_C \vec{E} \cdot d\vec{l} = 0 \qquad \boxed{\vec{E}_{1t} = \vec{E}_{2t}}$$

Divergence of $\vec{D}$:

$$\vec{\nabla} \cdot \vec{D} = \rho_V \longrightarrow \oint_S \vec{D} \cdot d\vec{s} = Q \qquad \boxed{\epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s}$$

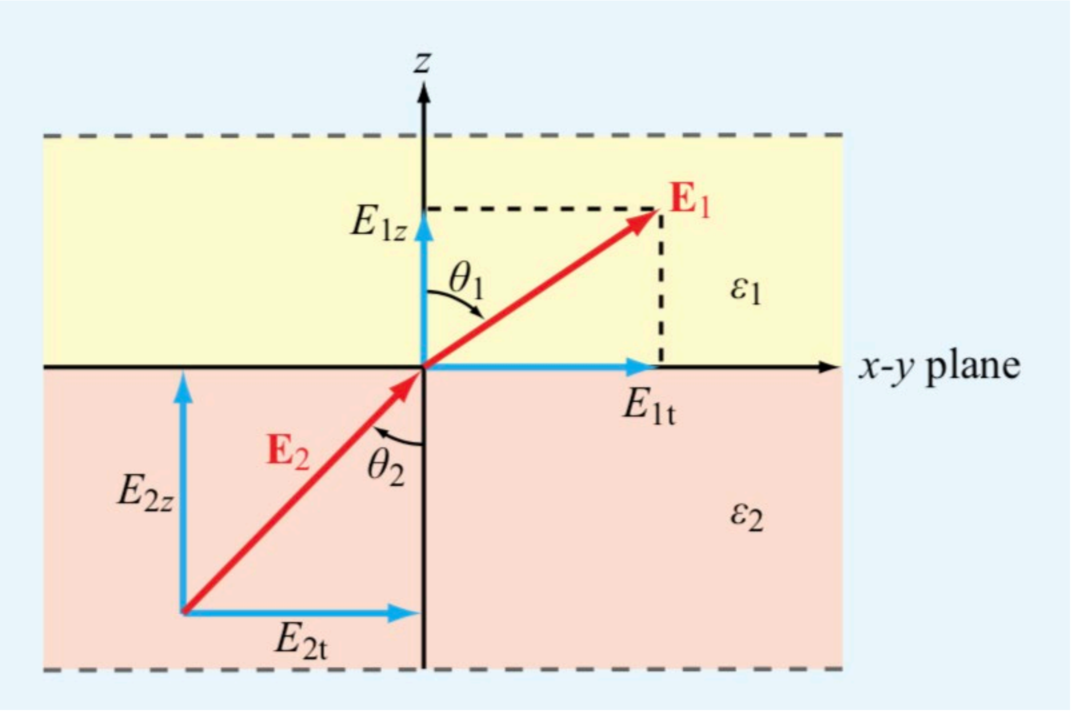
Summary of Boundary Conditions

Table 4-3: Boundary conditions for the electric fields.

Field Component	Any Two Media	Medium 1 Dielectric ε_1	Medium 2 Conductor
Tangential E	$\mathbf{E}_{1t} = \mathbf{E}_{2t}$	$\mathbf{E}_{1t} = \mathbf{E}_{2t} = 0$	
Tangential D	$\mathbf{D}_{1t}/\varepsilon_1 = \mathbf{D}_{2t}/\varepsilon_2$	$\mathbf{D}_{1t} = \mathbf{D}_{2t} = 0$	
Normal E	$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$	$E_{1n} = \rho_s/\varepsilon_1$	$E_{2n} = 0$
Normal D	$D_{1n} - D_{2n} = \rho_s$	$D_{1n} = \rho_s$	$D_{2n} = 0$

Notes: (1) ρ_s is the surface charge density at the boundary; (2) normal components of $\mathbf{E}_1$, $\mathbf{D}_1$, $\mathbf{E}_2$, and $\mathbf{D}_2$ are along $\hat{\mathbf{n}}_2$, the outward normal unit vector of medium 2.

Example: 2 dielectrics

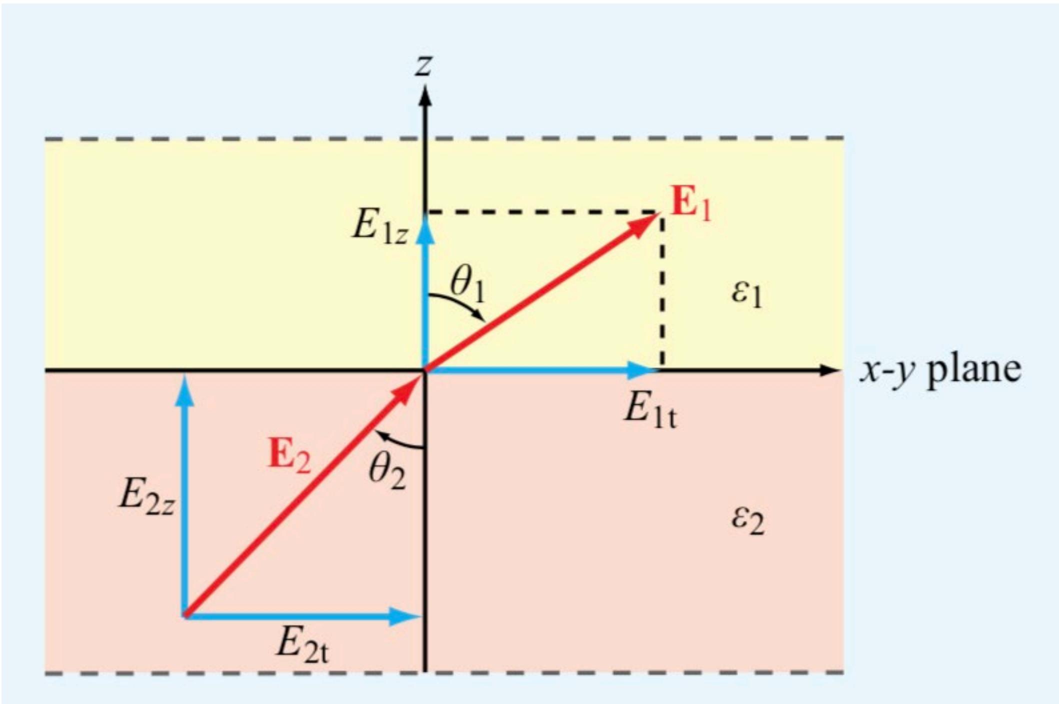


x-y plane boundary charge-free between 2 dielectrics, ϵ_1 and ϵ_2

$\vec{E}_1 = \hat{x}E_{1x} + \hat{y}E_{1y} + \hat{z}E_{1z}$
in dielectric ϵ_1

Find $\vec{E}_2$, θ_1, θ_2

Example: 2 dielectrics



x-y plane boundary charge-free between 2 dielectrics, ϵ_1 and ϵ_2

$$\boxed{\vec{E}_1 = \hat{x}E_{1x} + \hat{y}E_{1y} + \hat{z}E_{1z} \text{ in dielectric } \epsilon_1}$$

Find $\vec{E}_2, \theta_1, \theta_2$

Let $\vec{E}_2 = \hat{x}E_{2x} + \hat{y}E_{2y} + \hat{z}E_{2z}$ Normal to interface is $\hat{z}$, x-y are tangential

$E_{2x} = E_{1x}$ and $E_{2y} = E_{1y}$ tangential

$D_{2z} = D_{1z} \rightarrow \epsilon_2 E_{2z} = \epsilon_1 E_{1z}$ normal (charge free)

$$\boxed{\vec{E}_2 = \hat{x}E_{1x} + \hat{y}E_{1y} + \hat{z}\frac{\epsilon_1}{\epsilon_2}E_{1z}}$$

Example: 2 dielectrics

To obtain angles:

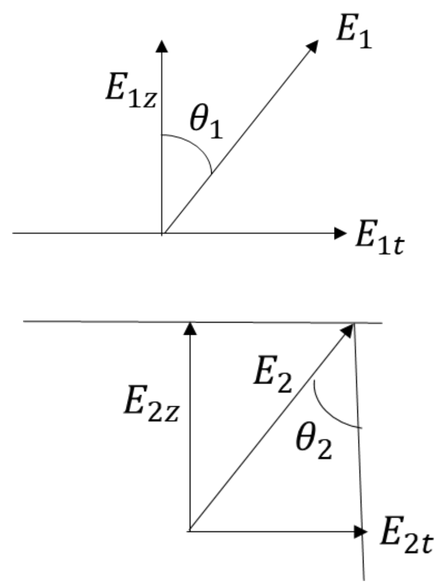
$$E_{1t} = \sqrt{{E_{1x}}^2 + {E_{1y}}^2}$$

$$E_{2t} = \sqrt{{E_{2x}}^2 + {E_{2y}}^2}$$

$$tan\theta_1 = \frac{E_{1t}}{E_{1z}} = \frac{\sqrt{{E_{1x}}^2 + {E_{1y}}^2}}{E_{1z}}$$

$$tan\theta_1 = \frac{E_{1t}}{E_{1z}} = \frac{\sqrt{{E_{1x}}^2 + {E_{1y}}^2}}{E_{1z}}$$

$$tan\theta_2 = \frac{E_{2t}}{E_{2z}} = \frac{\sqrt{{E_{2x}}^2 + {E_{2y}}^2}}{E_{2z}}$$



$$\frac{tan\theta_2}{tan\theta_1} = \frac{\epsilon_2}{\epsilon_1}$$

Dielectric – conductor boundary

Perfect conductor medium 2:

$$\vec{E}_2 = \vec{D}_2 \quad = 0$$

$$E_{1t} = D_{1t} = 0$$

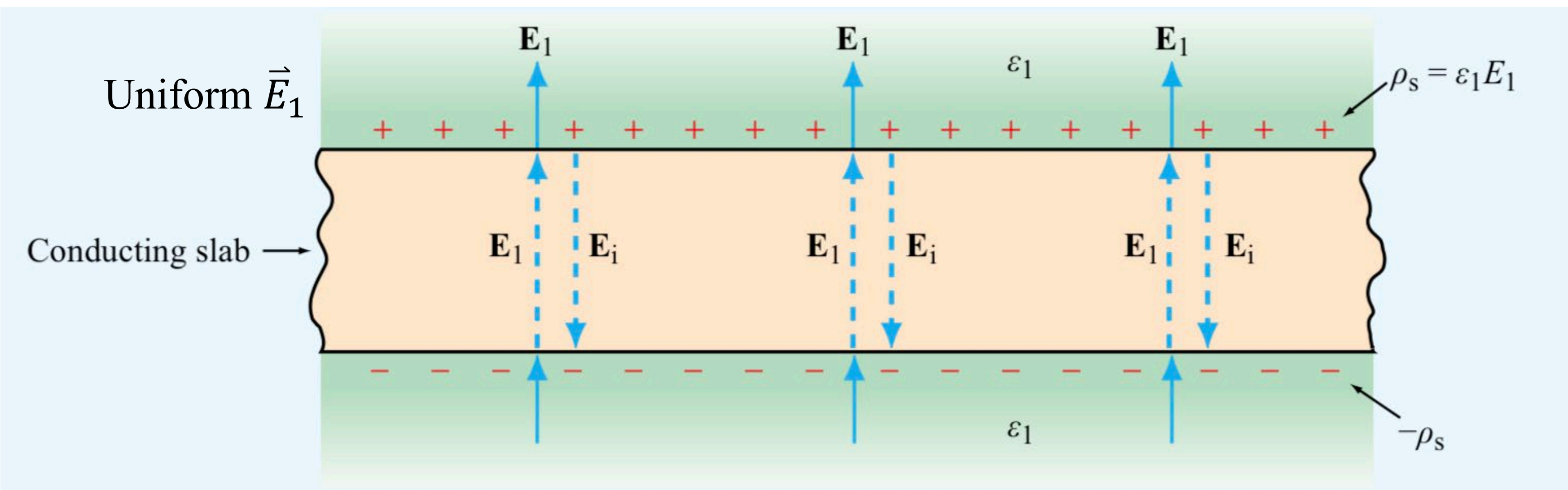
$$D_{1n} = \epsilon_1 E_{1n} \quad = \rho_s \longrightarrow \boxed{\vec{D}_1 = \epsilon_1 \vec{E}_1 = \hat{n}\rho_s}$$ At conductor surface

$\hat{n}$ from conductor

$\hat{n}$

Medium 1: ϵ_1

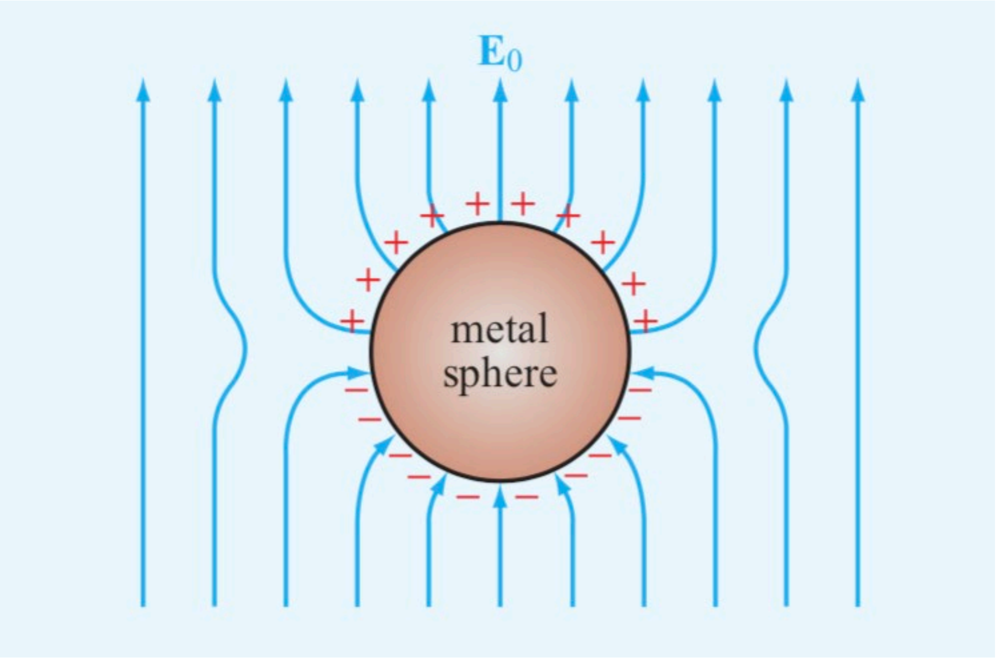
Medium 2: $\sigma = \infty$



$\vec{E}_i = -\vec{E}_1$ since field inside conductor <u>must = 0</u>

Net electric field inside a conductor is zero

Dielectric – conductor boundary



Metal in external field, $\vec{E}_0$

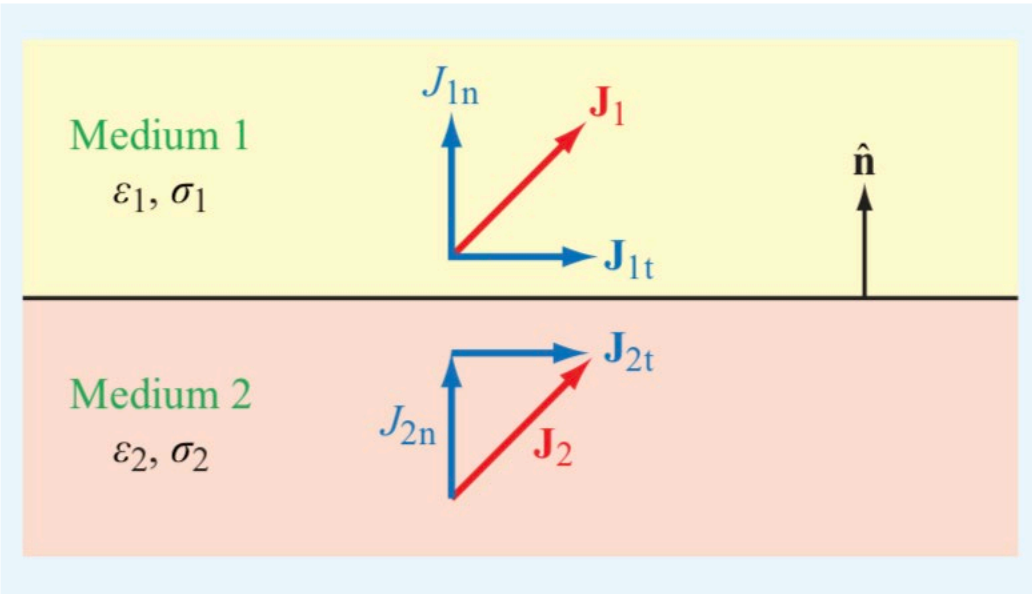
$\vec{E}$ points in = neg charge

$\vec{E}$ points out = pos charge

$\vec{E}$ always normal to conductor

Conductor – conductor boundary

General, not perfect dielectric or perfect conductor



$$\vec{E}_{1t} = \vec{E}_{2t}$$

$$\epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s$$

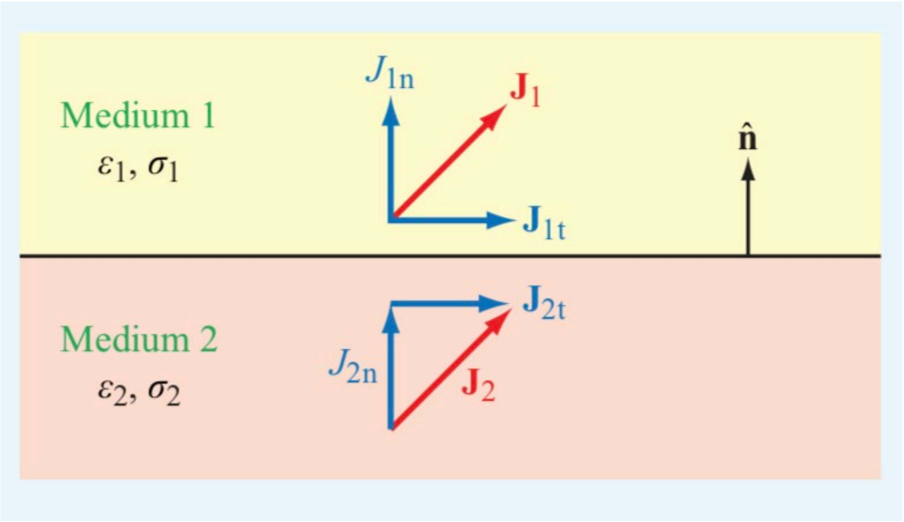
Electric field → gives rise to $\vec{J}_s$

$$\vec{J}_1 = \sigma_1 \vec{E}_1 \qquad \vec{J}_2 = \sigma_2 \vec{E}_2$$

$$E_{1n} = \frac{J_{1n}}{\sigma_1} \qquad E_{2n} = \frac{J_{2n}}{\sigma_2}$$

Conductor – conductor boundary

General, not perfect dielectric or perfect conductor



$$\vec{E}_{1t} = \vec{E}_{2t}$$

$$\epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s$$

Electric field → gives rise to $\vec{J}_s$

$$\vec{J}_1 = \sigma_1 \vec{E}_1 \qquad \vec{J}_2 = \sigma_2 \vec{E}_2$$

$$E_{1n} = \frac{J_{1n}}{\sigma_1} \qquad E_{2n} = \frac{J_{2n}}{\sigma_2}$$

$$\frac{\vec{J}_{1t}}{\sigma_1} = \frac{\vec{J}_{2t}}{\sigma_2} \qquad \epsilon_1\left(\frac{J_{1n}}{\sigma_1}\right) - \epsilon_2\left(\frac{J_{2n}}{\sigma_2}\right) = \rho_s$$

$\vec{J}_{1t}, \vec{J}_{2t}$ → current densities flowing in the 2 media parallel to boundary

If $J_{1n} \neq J_{2n}$ → then we have varying ρ_s at boundary and no longer electrostatics

<u>For electrostatics, $J_{1n} = J_{2n}$</u> $\qquad J_{1n}\left(\frac{\epsilon_1}{\sigma_1} - \frac{\epsilon_2}{\sigma_2}\right) = \rho_s$

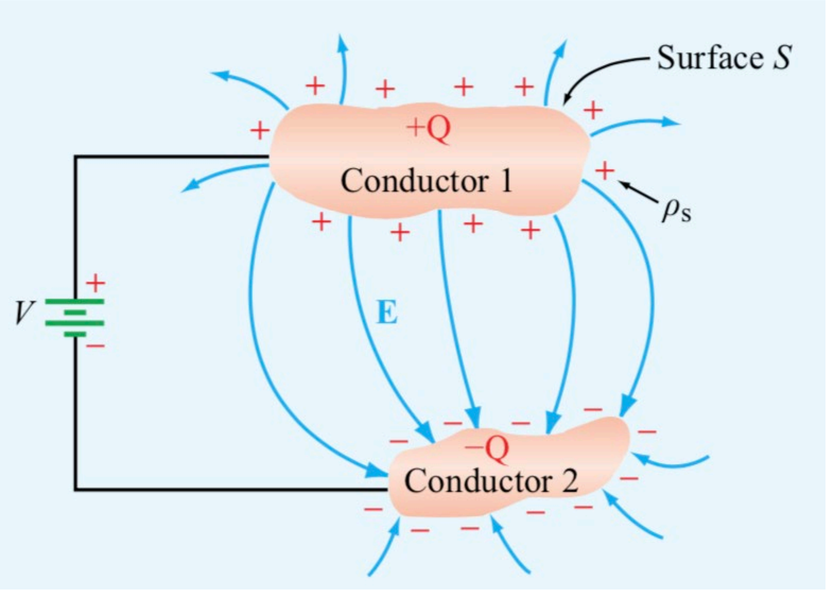
Capacitance

Separate 2 conductors by dielectric → forms a capacitor

Conductor excess charge always at surface distributed to maintain $\vec{E} = 0$ (equipotential) everywhere inside conductor

$$\text{Capacitance} = C = \frac{Q}{V} \qquad [\text{C/V or F}]$$

V: potential / voltage difference between conductors



$\vec{E} \rightarrow$ always E_n ($E_t = 0$ for conductors)

$$\boxed{E_n = \hat{n} \cdot \vec{E} = \frac{\rho_s}{\epsilon}}$$

Insulator between conductors

At conductor surface

Capacitance

$$Q = \int_S \rho_s ds = \int_S \epsilon \hat{n} \cdot \vec{E} ds = \int_S \epsilon \vec{E} \cdot d\vec{s}$$

$$V = V_{12} = -\int_{P_2}^{P_1} \vec{E} \cdot d\vec{l}$$

P_1, P_2 any points on conductors 1 and 2

$$C = \frac{Q}{V} = \frac{\int_S \epsilon \vec{E} \cdot d\vec{s}}{-\int_l \vec{E} \cdot d\vec{l}}$$

$S = +Q$ surface
P_1 is on S

Integration path from conductor 2 to 1

* Value of C is independent of E. It depends on geometry and materials

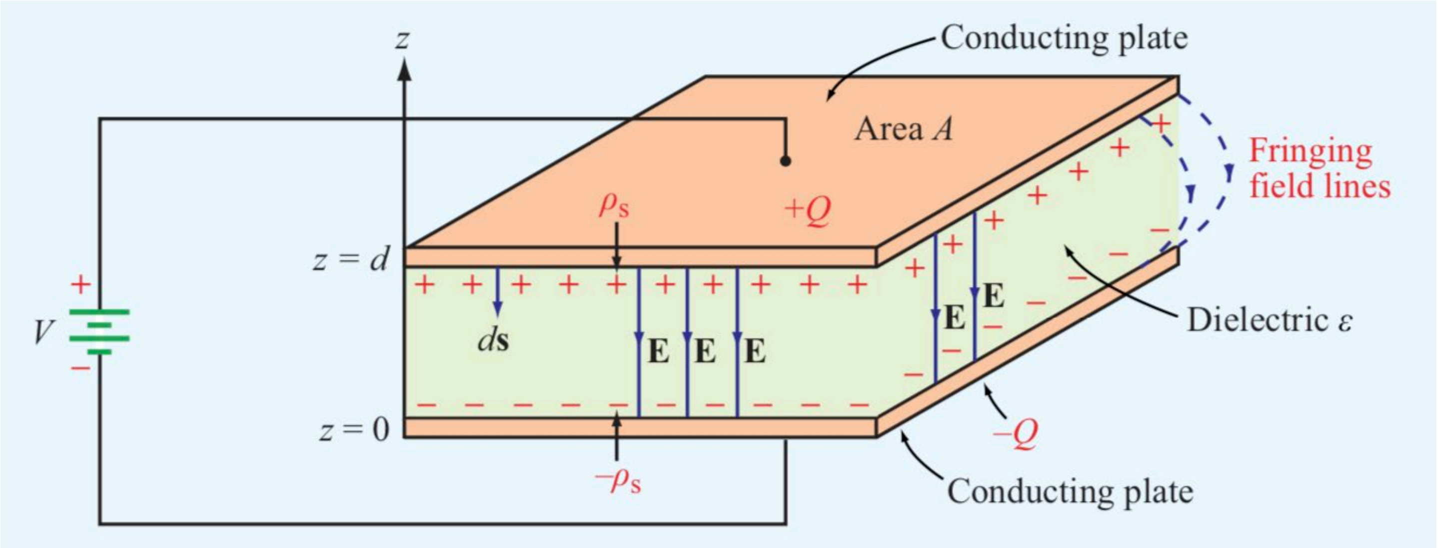
If dielectric is not perfect, can have some conductivity, resistance = R

$$R = \frac{V}{I} = \frac{-\int_l \vec{E} \cdot d\vec{l}}{\int_S \sigma \vec{E} \cdot d\vec{s}}$$

For medium with uniform σ, ϵ: RC = ϵ/σ

Parallel plate capacitor

2 parallel plates, surface area A separated by d filled with dielectric, ϵ.



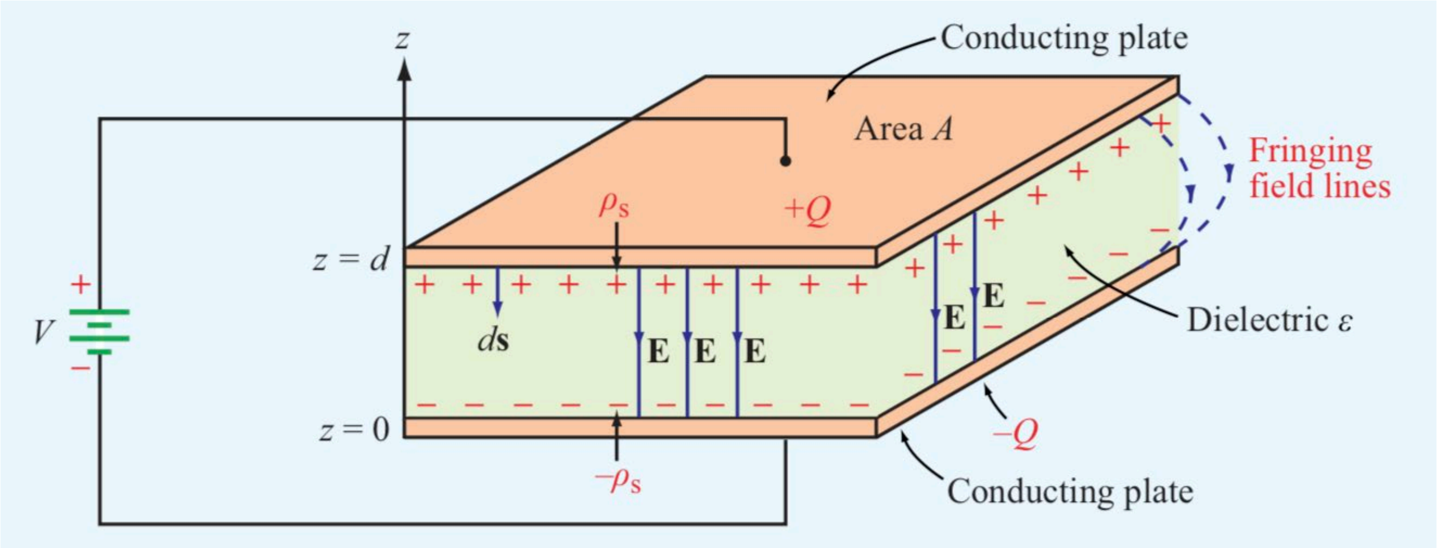
Ignore fringing fields

$\rho_s = \frac{Q}{A}$ upper plate $\qquad \vec{E} = -\hat{z}E$ (assume uniform)

$$E_n = \hat{n} \cdot \vec{E} = \frac{\rho_s}{\epsilon} \quad \Rightarrow \quad E_n = \frac{\rho_s}{\epsilon} = \frac{Q}{\epsilon A}$$

Parallel plate capacitor

2 parallel plates, surface area A separated by d filled with dielectric, ϵ.



Ignore fringing fields

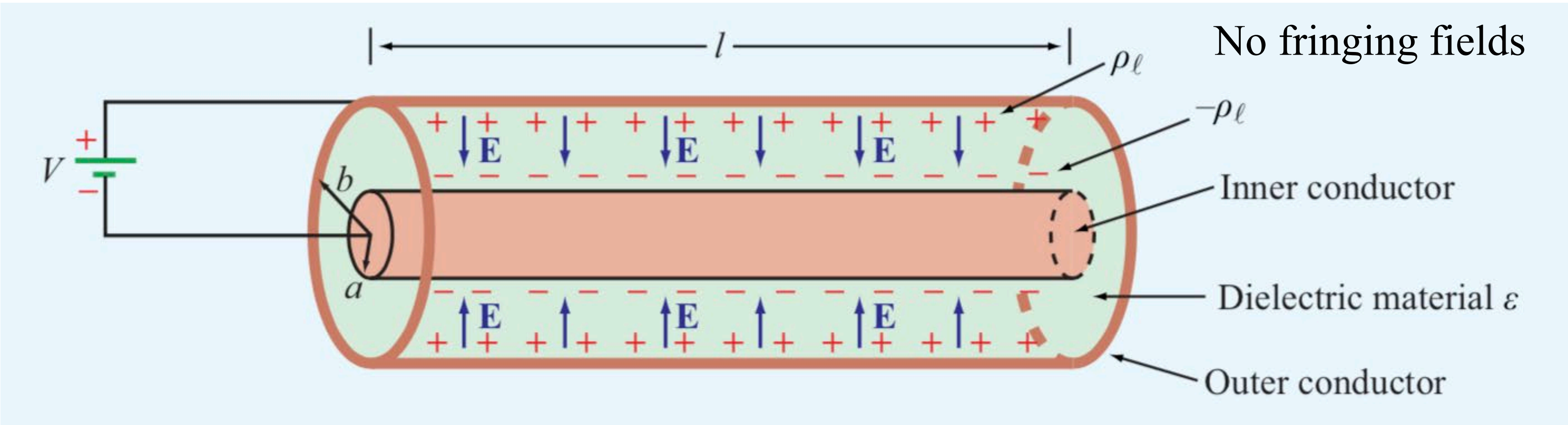
$\rho_s = \frac{Q}{A}$ upper plate $\vec{E} = -\hat{z}E$ (assume uniform) $E_n = \frac{\rho_s}{\epsilon} = \frac{Q}{\epsilon A}$

$$V = -\int_0^d \vec{E} \cdot d\vec{l} = -\int_0^d (-\hat{z}E) \cdot \hat{z}dz = Ed \qquad E = \frac{Q}{\epsilon A}$$

$$C = \frac{Q}{V} = \frac{Q}{Ed} = \frac{\epsilon A}{d}$$

C is independent of E. It depends on geometry and materials

Coax capacitance



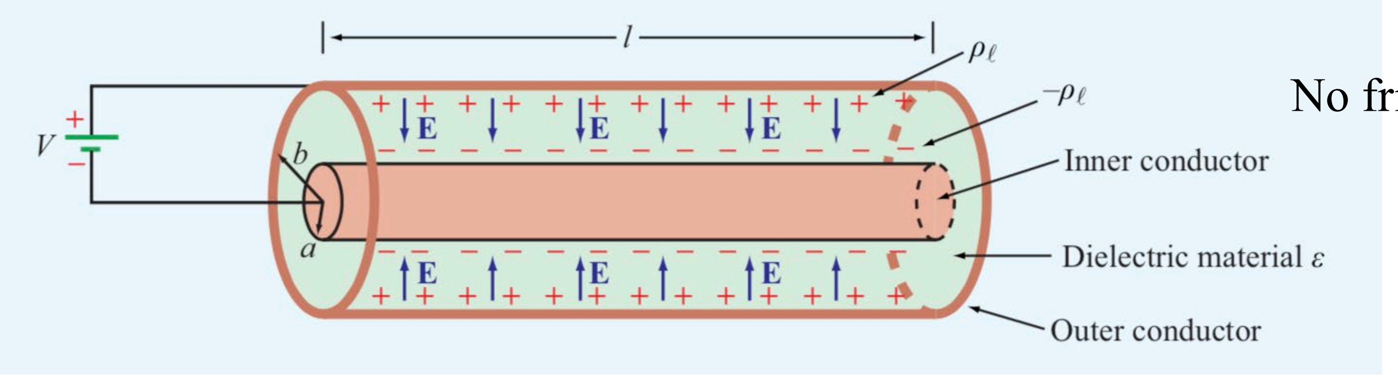
Given voltage, V

$+Q$ accumulates on outer conductor $-Q$ accumulates on inner conductor

Assume uniform charge distributions: (outer) $\rho_s' = \frac{Q}{2\pi bl}$ (inner) $\rho_s'' = \frac{-Q}{2\pi al}$

E_n

Coax capacitance



No fringing fields

Cylindrical Gaussian surface: $a < r < b$

$$\oint_S \vec{D} \cdot d\vec{s} = Q$$

$$\oint_S \epsilon \underset{E_r}{\vec{E}} \cdot \underset{2\pi r l}{d\vec{s}} = Q$$

$\vec{E}$ in $\hat{r}$ direction, points inward: $\vec{E} = -\hat{r}\dfrac{Q}{2\pi\epsilon r l}$

Potential between outer and inner:

$$V = -\int_a^b \vec{E} \cdot d\vec{l} = -\int_a^b \left(-\hat{r}\frac{Q}{2\pi\epsilon r l}\right) \cdot (\hat{r}\,dr) = \frac{Q}{2\pi\epsilon l}\ln\left(\frac{b}{a}\right)$$

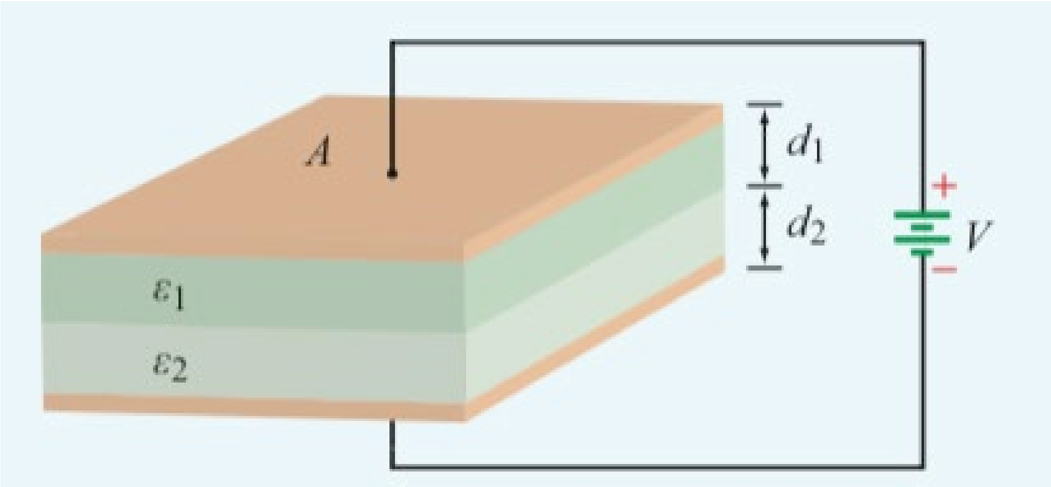
$$C = \frac{Q}{V} = \frac{2\pi\epsilon l}{\ln\left(\frac{b}{a}\right)}$$

Table 2.1 → $C' = \frac{C}{l} = \frac{2\pi\epsilon}{\ln\left(\frac{b}{a}\right)}$

E_n

Parallel plate capacitor

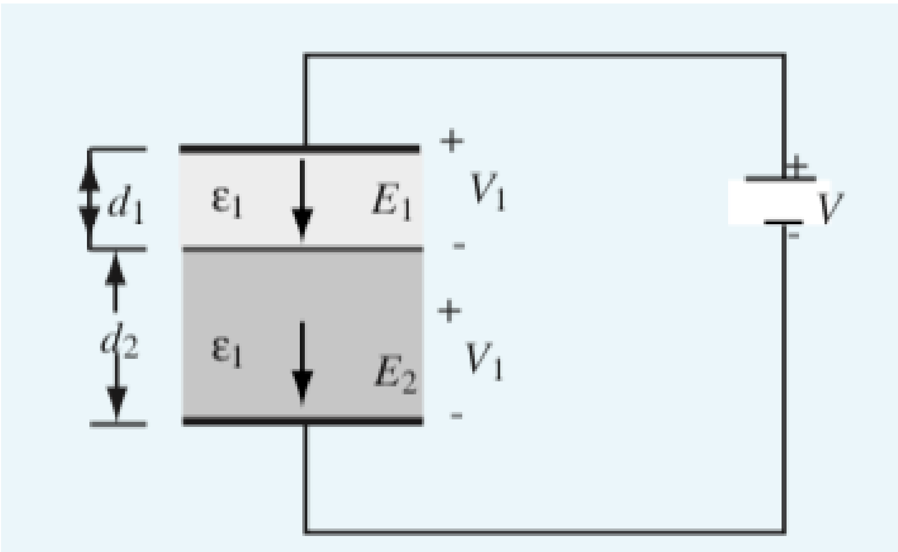
The capacitor shown in the following figure consists of two parallel dielectric layers.



V_1 and V_2 are the electric potentials across the upper and lower dielectrics, respectively.

<u>Obtain:</u> expressions for E_1 and E_2 in terms of ε_1 , ε_2 , V , and the indicated dimensions of the capacitor.

Parallel plate capacitor



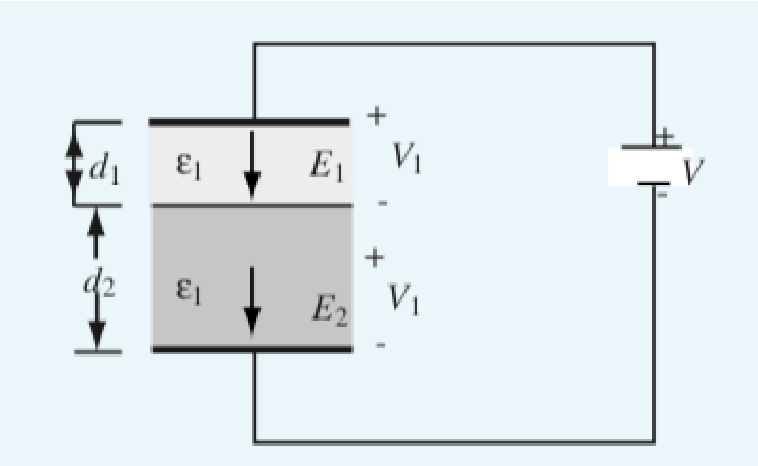
If V_1 is the voltage across the top layer and V_2 across the bottom layer, then:

$$V = V_1 + V_2$$

$$E_1 = \frac{V_1}{d_1} \qquad E_2 = \frac{V_2}{d_2}$$

Boundary conditions → the normal component of $\vec{\boldsymbol{D}}$ is continuous across the boundary (in the absence of surface charge).

Parallel plate capacitor



$$E_1 = \frac{V_1}{d_1}$$

$$E_2 = \frac{V_2}{d_2}$$

Boundary conditions → the normal component of $\vec{\boldsymbol{D}}$ is continuous

Electric fields inside of capacitor:

$$D_{1n} = D_{2n} \rightarrow \varepsilon_1 E_1 = \varepsilon_2 E_2$$

$$V = V_1 + V_2 = E_1 d_1 + E_2 d_2 = E_1 d_1 + \frac{\varepsilon_1 E_1}{\varepsilon_2} d_2$$

Solve for E_1 and E_2: $$E_1 = \frac{V}{d_1 + \frac{\varepsilon_1}{\varepsilon_2} d_2} \qquad E_2 = \frac{V}{d_2 + \frac{\varepsilon_2}{\varepsilon_1} d_1}$$